

Lipschitz Shadowing and Structural Stability

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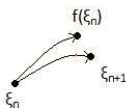
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Standard Shadowing and Structural Stability

- $f : M \rightarrow M$, $f \in C^1$, $M \in C^\infty$, dist.
- $\{\xi_n\}$ is d -pseudotrajectory, if $\text{dist}(\xi_{n+1}, f(\xi_n)) < d$



- Standard Shadowing (StSh)
 $\forall \varepsilon > 0 \exists d > 0$ such that $\forall d$ -pseudotrajectory $\{\xi_n\}$ there exists exact trajectory $\{x_n\}$ such that

$$\text{dist}(x_n, \xi_n) < \varepsilon.$$

- SS – set of structurally stable diffeomorphisms.
 There exists neighborhood U_f in the C^1 -topology such that for any $g \in U_f$, diffeomorphisms f and g are topologically conjugated.

Known Facts and Hypothesis

- $SS \subset StSh$ (Robinson 1977, Sawada 1980).
 - Shadowing lemma: If Λ is hyperbolic then f has shadowing in some $U(\Lambda)$.
- $SS \neq StSh$.
- $Int^1(StSh) = SS$ (Sakai, 1994).
- Hypothesis Abdenur-Diaz: generically $StSh = SS$.
- Shadowing $?\Rightarrow?$ Structural Stability.

Lipschitz Shadowing

- Lipschitz Shadowing (LipSh)
 $\exists L, d_0 > 0$ such that $\forall d < d_0$ and d -pseudotrajectory $\{\xi_n\}$
 there exists exact trajectory $\{x_n\}$ such that

$$\text{dist}(x_n, \xi_n) < Ld.$$

Theorem (Pilyugin, Tikhomirov, 2009)

SS = LipSh.

- Expansivity (EXP): $\exists a > 0$ such that if $\forall n \in \mathbf{Z}$
 $\text{dist}(f^n(x), f^n(y)) < a$ then $x = y$.
- $\text{SS} \cap \text{EXP} = \text{Anosov}$ (Mane, 1974).
 - Anosov diffeomorphism – whole manifold is a hyperbolic set.

Consequence

$\text{LipSh} \cap \text{EXP} = \text{Anosov}$.

Periodic Shadowing

- ΩS – set of Ω -stable diffeomorphisms
 There exists neighborhood U_f in the C^1 -topology such that for any $g \in U_f$, f is topologically conjugated to g on $\Omega(f)$.
- Periodic Shadowing (PerSh)
 $\forall \varepsilon > 0 \exists d > 0$ such that \forall periodic d -pseudotrajectory $\{\xi_n\}$
 there exists periodic exact trajectory $\{x_n\}$ such that

$$\text{dist}(x_n, \xi_n) < \varepsilon.$$
- Lipschitz Periodic Shadowing (LipPerSh)
 $\varepsilon = Ld.$

Theorems (Osipov, Pilyugin, Tikhomirov, 2009)

- $\text{LipPerSh} = \Omega S.$
- $\text{Int}^1(\text{PerSh}) = \Omega S.$

Proof of LipSh = SS

Mane, 1977:

$$E^s(x) = \{v \in T_x M, Df^n(x)v \rightarrow_{n \rightarrow +\infty} 0\}$$

$$E^u(x) = \{v \in T_x M, Df^n(x)v \rightarrow_{n \rightarrow -\infty} 0\}$$

Theorem: If $\forall x \in M \quad E^s(x) \oplus E^u(x) = T_x M$ then $f \in SS$.

Pliss, 1977:

p_n - exact trajectory, $A_n = Df(p_n)$.

$$v_{n+1} = A_n v_n + w_n.$$

Theorem: If $\forall |w_n| < 1$ there exists $|v_n| < \infty$ then

$$E^s(p_n) \oplus E^u(p_n) = T_{p_n} M.$$

Proof of $\text{LipPerSh} = \Omega S$

- $f \in \text{LipPerSh}$, Lipschitz constant $L > 0$.
- Periodic orbits are hyperbolic.
- Periodic orbits are uniformly hyperbolic
 - p_n – periodic trajectory, $v \in E^u(p_0)$.
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$$|Df^k(p_0)v| \geq \frac{1}{L} \left(1 + \frac{1}{L}\right)^{k-1}, \quad k > 0.$$

- Periodic orbits are dense in $\Omega(f)$.
- Passing to a limit we prove Axiom A.
- “no-cycle” condition.

Conclusion

Theorems

- LipSh = SS.
- LipPerSh = Ω S.

Conclusion

Lipschitz Shadowing \Rightarrow Hyperbolicity.

Main Idea

$$f(x) \leftrightarrow v_{n+1} = A_n v_n.$$

Thank you very much for your attention!